JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 31, No. 3, May–June 2008

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Survey of Orbit Element Sets

Gerald R. Hintz*
University of Southern California,
Los Angeles, California 90089

DOI: 10.2514/1.32237

Nomenclature

a = semimajor axis
e = eccentricity
h = second equinoctial element
I = retrograde factor
i = inclination
k = third equinoctial element
L = true longitude
M = mean anomaly
θ = true anomaly
λ = mean longitude
Ω = right ascension of ascending node
ω = argument of periapsis

Introduction

A SURVEY of orbit element sets was conducted as part of a research and technology development task [1] which sought to develop feedback control laws and prototype tools for computing feasible, low-thrust transfers in multibody environments, where at least two celestial bodies exert significant gravitational influence on the spacecraft. One approach considered in this research effort was to use two-body element sets and their associated variational equations with the multibody effects modeled as perturbations. The goal of the survey was to identify an orbit element set that is the most natural to use in developing feedback control laws. This survey identified 22 candidate orbit element sets plus variations defined in terms of Euler angles, Euler parameters, functions of classical elements, quaternions, set-III elements, fast or slow variables, or canonical variables. The list of candidates is provided in this paper, together with a brief description of their attributes.

Orbit Element Sets

The approach followed was to survey the literature to identify the candidate orbit element sets (OES) and to select the best of those available. The candidates are listed in Table 1. One objective of this search was to find a nonsingular OES. Singularities cause rapid oscillations in orbit elements when the orbit is near a singular point [4] and, therefore, lead to difficulties in integrating the equations of motion (EOM) expressed in those elements.

The satellite equations of motion can be expressed in various forms. A simple Cartesian form [Table 1 (a)] is often used for numerical integration. However, other forms based on the variation of parameters are often used in perturbation analyses.

The analysis of Keplerian motion is most often formulated in terms of classical orbit elements [Table 1(b) and (c)]. When perturbations are introduced, time variation of these elements can be calculated based on a standard variation of parameters procedure. Euler angles are used to parameterize the orientation of the orbit plane. However, due to the inherent singularities of the Euler angles, the variational equations may become singular for zero eccentricity and/or zero inclination because Ω and ω are indeterminate for $i = 0$ or π and ω is indeterminate for $e = 0$. Other sets [Table 1 (d)–(f)] have been introduced that have polar singularities.

Equinoctial elements [Table 1(g)–(j)] have been introduced [5,6] to remove these singularities. This set of elements contains h and k [f and g for the modified set (Table 1(k)], which are functions of the classical elements e, ω, and Ω and are the components in the orbital frame of the eccentricity vector that points toward the periapsis and has magnitude e. The parameters p and q (h and k for the modified set), which are functions of the classical orientation angles i and Ω, are required in the rotation matrix between the inertial frame and the orbital elements. The element λ is the mean longitude as used in the classical literature. There is a singularity at $i = \pi$, which can be removed by defining the p_r and q_r elements for retrograde orbits in terms of the cotangent function and substituting $-\Omega$ for Ω in the h_r, k_r, and λ_r elements [4] as shown in Table 1(g) and (k). Note that $p = p_r$ and $q = q_r$ at $i = \pi/2$, where the posigrade and retrograde elements are patched together. But it is not necessary to switch between these element sets during a numerical integration when the value of i passes $\pi/2$. This "switching" value is only a guideline for choosing the element set at the beginning of the integration to avoid the singularity at $i = \pi$ for posigrade elements and $i = 0$ for retrograde elements [20]. For the retrograde equinoctial elements, the quantities h_r and k_r are the components of the eccentricity vector relative to the retrograde orbital frame. An example of using the equinoctial elements in optimization is given in [21]. Lagrange introduced this element set using $\tan(i)$ instead of $\tan(i/2)$ [Table 1 (h)] in 1774 for his study of secular variations [5]. This replacement introduces a singularity at the poles.

However, the equinoctial elements are singular for parabolic orbits, so modified equinoctial elements [Table 1(k)] have been introduced to remove this singularity by replacing the semimajor axis with the semilatus rectum. The modified set also replaces the mean longitude λ with the true longitude L as the element fixing position in the orbit [12,13], avoiding the use of the elliptic parameter M.

Astronomers and physicists prefer the Kustaanheimo–Stiefel (KS) orbital elements [Table 1(l)]. The KS elements [14] have been shown to form a quaternion, comprising complex functions of both orbital elements and the inertial position and velocity. These parameters have little physical meaning [15].

A nonsingular (for $e < 1$) quaternion element set [Table 1(m)] is introduced in [11]. At each of the classical singularities for $e < 1$, that is, $e = 0$, $i = 0$, and $i = \pi$, one pair of the elements has as a factor the sine or cosine of an indeterminate angle, but another factor [e, $\sin(i/2)$, or $\cos(i/2)$] is zero. The q elements form a quaternion. This reference also mentions a canonical element set [Table 1(n)].

Presented as Paper 30 at the AIAA Southern California Aerospace Systems and Technology Conference, Santa Ana, California, 5 May 2007; received 21 May 2007; accepted for publication 26 October 2007.

*Adjunct Professor, Astronautics and Space Technology Division, 854 Downey Way/Mail Station 1192. Associate Fellow AIAA.

Table 1 Orbit element sets

(a)	Cartesian (inertial) coordinates: x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$	
(b)	True classical element set [2]: a (semimajor axis) e (eccentricity) i (inclination) ω (argument of periapsis) Ω (right ascension of the ascending node) time of perifocal passage Singular at $e = 0$ or 1, $i = 0$ or π.	
(c)	Modified classical element set [2]: a, e, i, ω, Ω, and θ or $M = nt_p$ or t_p or $\omega + \theta$ (argument of latitude), where $\theta =$ true anomaly, $M =$ mean anomaly, $n =$ mean motion, and $t_p =$ time after periapsis passage. Preferred over the true classical element set to avoid complications in computing the time of perifocal passage. Singular at $e = 0$ or 1, $i = 0$ or π.	
(d)	ADBARV elements [3]: RA (right ascension) DEC (declination) β (flight direction angle measured from vertical) AZ (azimuth) r (distance from center of attraction) v (magnitude of inertial velocity vector) Singular at the poles The name ADBARV of this element set is derived from the names of the elements in the set.	
(e)	Geographic (GEO) elements [3]: λ (longitude) ϕ (latitude) β AZ r v Singular at the poles	
(f)	Spherical coordinates [3]: r RA DEC v α_v (right ascension of velocity vector) δ_v (declination of velocity vector) Singular at the poles	
(g)	Equinoctial elements: For posigrade orbits [3–7] a $h = e * \sin(\omega + \Omega)$ $k = e * \cos(\omega + \Omega)$ $p = \tan(i/2) * \sin\Omega$ $q = \tan(i/2) * \cos\Omega$ $\lambda = M + \omega + \Omega$ (mean longitude) Undefined for $e \geq 1$.	 For retrograde orbits [4] a $h_r = e * \sin(\omega - \Omega)$ $k_r = e * \cos(\omega - \Omega)$ $p_r = \cot(i/2) * \sin\Omega$ $q_r = \cot(i/2) * \cos\Omega$ $\lambda_r = M + w - \Omega$
(h)	Another version of this set replaces $\tan(i/2)$ by $\tan(i)$ ([8], p. 288), but this replacement introduces a singularity for polar orbits.	
(i)	Alternate equinoctial elements [9,10]: a $\xi = e * \sin(\omega + \Omega)$ $\eta = e * \cos(\omega + \Omega)$ $\zeta = \sin(i/2) * \sin\Omega$ $\nu = \sin(i/2) * \cos\Omega$ $\lambda = M + \omega + \Omega$ Undefined for $e \geq 1$.	
(j)	Another version [11] of this set replaces $\sin(i/2)$ by $\sin(i)$.	
(k)	Modified equinoctial elements [12,13]: p (semilatus rectum) $f = e * \cos(\omega + I\Omega)$ $g = e * \sin(\omega + I\Omega)$ $h = \tan^I(i/2) * \cos\Omega$ $k = \tan^I(i/2) * \sin\Omega$ $L = \omega + I\Omega + \theta$ (true longitude), where the retrograde factor $I = +1$ for posigrade orbits and $I = -1$ for retrograde orbits and θ denotes the true anomaly. Nonsingular for all eccentricities and inclinations	
(l)	Kustaanheimo–Stiefel elements—see [14].	

(Continued)

Table 1 Orbit element sets(Continued)

(m) Nonsingular ($e < 1$) quaternion element set [11]:
$q_0 = p^{1/4}\cos(i/2)\cos((\Omega + \omega + M)/2)$
$q_1 = p^{1/4}\sin(i/2)\cos((\Omega - \omega - M)/2)$
$q_2 = p^{1/4}\sin(i/2)\sin((\Omega - \omega - M)/2)$
$q_3 = p^{1/4}\cos(i/2)\sin((\Omega + \omega + M)/2)$
$e_X = e\cos M$
$e_Y = -e\sin M$
Nonsingular for $e < 1$.

(n) Nonsingular ($e < 1$) canonical elements [11]:
$\xi_1 = [2(G + H)]^{1/2}\cos((\omega + M + \Omega)/2)$
$\xi_2 = [2(G - H)]^{1/2}\cos((\omega + M - \Omega)/2)$
$\xi_3 = [2(L - G)]^{1/2}\cos M$
$\eta_1 = [2(G + H)]^{1/2}\sin((\omega + M + \Omega)/2)$
$\eta_2 = [2(G - H)]^{1/2}\sin((\omega + M - \Omega)/2)$
$\eta_3 = [2(L - G)]^{1/2}\sin M$
where $L = (\mu a)^{1/2}$, $G = (\mu p)^{1/2}$ and $H = (\mu p)^{1/2}\cos i$ are Delaunay canonical variables.
Undefined for $e \geq 1$.

(o) Euler parameter set [15]:
a
$\eta = \sqrt{1 - e^2}$ (the eccentricity measure)
$\varepsilon_1 = \sin(\phi/2)\cos\phi_1$
$\varepsilon_2 = \sin(\phi/2)\cos\phi_2$
$\varepsilon_3 = \sin(\phi/2)\cos\phi_3$
M (mean anomaly),
where ϕ is the rotation angle about the Euler vector and ϕ_i is the angle between the Euler vector and the inertial unit vectors.
Undefined for $e \geq 1$.

(p) Same as item (o), except e replaces the eccentricity measure η ([15]).
Singular at $e = 0$ and 1.

(q) Delaunay elements [16]:
M
ω
Ω
$L = \sqrt{\mu a}$
h (angular momentum)
$H = \sqrt{\mu p}(\cos(i))$
Canonical counterpart to classical, or Keplerian, elements with the same singularities.

(r) Poincaré elements [17,18]:
$\Lambda = \sqrt{\mu a}$
$\xi = e * \sin(\omega + \Omega)\sqrt{2\Lambda/(1 + \sqrt{1 - e^2})}$
$\eta = e * \cos(\omega + \Omega)\sqrt{2\Lambda/(1 + \sqrt{1 - e^2})}$
$u = \sin(i)\sin(\Omega)\sqrt{2\Lambda\sqrt{1 - e^2}/(1 + \cos(i))}$
$v = \sin(i)\cos(\Omega)\sqrt{2\Lambda\sqrt{1 - e^2}/(1 + \cos(i))}$
$\lambda = M + \omega + \Omega$
The set of Poincaré elements can be expressed in terms of the Keplerian and equinoctial variables as follows:
$\Lambda = \sqrt{\mu a}$
$\xi = h\sqrt{2\Lambda/(1 + \sqrt{1 - e^2})}$
$\eta = k\sqrt{2\Lambda/(1 + \sqrt{1 - e^2})}$
$u = p\sqrt{2\Lambda\sqrt{1 - e^2}/(1 + \cos(i))}$
$v = q\sqrt{2\Lambda\sqrt{1 - e^2}/(1 + \cos(i))}$
$\lambda = M + \omega + \Omega$
Canonical counterpart to equinoctial elements
Undefined for $e \geq 1$

(s) Alternate form of Poincaré elements [16]:
$L = \sqrt{\mu a}$
$g_p = \sqrt{2L(1 - \sqrt{1 - e^2})}\cos(\omega + \Omega)$
$h_p = \sqrt{2L\sqrt{1 - e^2}(1 - \cos(i))}\cos(\Omega)$
$G = \sqrt{2L(1 - \sqrt{1 - e^2})}\sin(\omega + \Omega)$
$H = \sqrt{2L\sqrt{1 - e^2}(1 - \cos(i))}\sin(\Omega)$
$\lambda = M + \omega + \Omega$,
where L, G, and H denote the Delaunay canonical variables.
Canonical counterpart to equinoctial elements.
Undefined for $e \geq 1$.

(t) Set-III elements—see [8], p. 241.

(u) Kamel's fast variables—see [19], p. 388, Eq. (12).

(v) Kamel's slow variables—see [19], p. 388, Eq. (19).

[Canonical variables have special characteristics and relationships which make them useful for astrodynamics. The equations of motion for two-body motion can be expressed as the product of a 6×6 matrix times the 6×1 state vector containing the position and velocity vectors. Using standard elements, the 6×6 matrix contains many nonzero terms on and off the diagonal. Using canonical variables, the 6×6 matrix is diagonal, which simplifies the construction of perturbation solutions (see [16], Chaps. 2 and 9).]

Because the singularity of the variational equations for classical elements stems from the use of orbital angles, Gurfil [15] replaced the Euler angles with Euler parameters [Table 1(o), (p)]. These Euler parameters also form a quaternion, but, unlike the KS elements, the Euler parameters are well-defined geometric quantities. Gurfil developed the planetary equations in terms of Euler parameters, replacing the inclination, argument of periapsis, and right ascension of the ascending node without resorting to quaternion algebra. The resulting variational equations are expressions in Lagrange form.

The Delaunay elements [Table 1(q)] are the canonical counterpart to the classical, or Keplerian, orbital elements. As such, they contain singularities for small values of eccentricity and inclination [16].

The Poincaré elements [Table 1(r)] are the canonical counterpart of the equinoctials, so they are nonsingular for small eccentricities and inclinations. They can also be defined in terms of the Keplerian and equinoctial elements [17,18] [Table 1(r)]. An alternate form [Table 1(s)] is considered in [16].

The equinoctials are similar to the set-III elements [8] [Table 1(t)]. However, the equinoctials are defined by finite explicit relations, while the set-III elements are defined by nonintegrable differential relations. Also, the inverse partial derivatives for the set-III elements do not exist [6] for $e = 0$.

Kamel [19] describes an element set [Table 1(u)] in which the orbit eccentricity, true anomaly, argument of latitude $(\omega + \theta)$, and orbit inclination are replaced by four fast variables whose unperturbed periods are essentially equal to the orbital period. Kamel also introduces another set [Table 1(v)] in which the four fast variables are transformed to four slow variables such that the eccentricity vector variation is independent of the out-of-plane acceleration.

Equations of Motion

Lagrange's planetary equations ([5], p. 483) describe the time rate of change of the classical orbit elements under an arbitrary perturbing potential. Thus, the evolution of a satellite orbit under perturbing forces is described by a system of six differential equations, representing the rate of change of a set of variables that describes the orbit state.

However, the conventional Lagrange planetary equations are not the most suitable perturbation equations for many practical applications. The presence of e, $(1 - e^2)^{1/2}$, and $\sin(i)$ in the denominators of some of the variation of parameters (VOP) equations produces singularities for $e = 0$ or 1 and $i = 0$ or π that may lead to difficulties in the integration of the equations of motion. Furthermore, the osculating (true, mean, and eccentric) anomalies become ill defined for small eccentricity [22]. The practical application of the VOP equations of motion requires that the element set be selected carefully for the range of application.

Rationale for selecting the equinoctial elements to eliminate singularities in the EOM is given in [5], Sec. 10.4. The VOP equations of motion for equinoctial elements are stated concisely in [20] for both posigrade (direct) and retrograde orbits with the perturbations given by a disturbing function R. The VOP equations are given in [4] for third-body, J_2, J_3, and (nonconservative) atmospheric drag [Eqs. (61)–(66)] perturbations.

The EOM for the modified equinoctial elements are nonsingular for all eccentricities and inclinations as shown in Table 2. Note that the components of the perturbing force must be expressed in terms of the set of elements adopted. In the Gauss formulation of the EOM, the perturbing forces are expanded in three local components along radial, transverse, and orbit-normal directions. The Gauss formulation for classical elements is given in [5] and the Gauss formulation for modified equinoctial elements is given in Table 2. Table 3 identifies where the Lagrange and Gauss forms of the VOP equations are available for a selected subset of the elements defined in Table 1. For those same orbit element sets, Table 4 shows if and where the conversion between the orbital elements and position and velocity vectors in Cartesian coordinates can be found.

If initial conditions are specified for the orbit elements, the differential equations can be integrated by any convenient numerical method. Generally, when the disturbing acceleration is small, a relatively large integration step can be employed. However, in the neighborhood of a singularity, the rates of change of orbit elements can be large, despite the fact that the disturbing acceleration is small [5].

Table 2 Equations of motion for modified equinoctial elements

The equations of motion [12,13] are given for posigrade orbits using auxiliary (positive) variables.

$s^2 = 1 + h^2 + k^2$ and $w = p/r = 1 + f * \cos(L) + g * \sin(L)$

Lagrange variational equations:

$$\frac{dp}{dt} = 2\sqrt{\frac{p}{\mu}}\left(-g\frac{\partial R}{\partial f} + f\frac{\partial R}{\partial g} + \frac{\partial R}{\partial L}\right)$$

$$\frac{df}{dt} = \frac{1}{\sqrt{\mu p}}\left\{2pq\frac{\partial R}{\partial p} - (1 - f^2 - g^2)\frac{\partial R}{\partial g} - \frac{gs^2}{2}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right) + [f + (1 + w)\cos(L)]\frac{\partial R}{\partial L}\right\}$$

$$\frac{dg}{dt} = \frac{1}{\sqrt{\mu p}}\left\{-2pf\frac{\partial R}{\partial p} + (1 - f^2 - g^2)\frac{\partial R}{\partial f} + \frac{fs^2}{2}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right) + [g + (1 + w)\sin(L)]\frac{\partial R}{\partial L}\right\}$$

$$\frac{dh}{dt} = \frac{s^2}{2\sqrt{\mu p}}\left\{h\left(g\frac{\partial R}{\partial f} - f\frac{\partial R}{\partial g} - \frac{\partial R}{\partial L}\right) - \frac{s^2}{2}\frac{\partial R}{\partial k}\right\}$$

$$\frac{dk}{dt} = \frac{s^2}{2\sqrt{\mu p}}\left\{k\left(g\frac{\partial R}{\partial f} - f\frac{\partial R}{\partial g} - \frac{\partial R}{\partial L}\right) + \frac{s^2}{2}\frac{\partial R}{\partial h}\right\}$$

$$\frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \frac{s^2}{2\sqrt{\mu p}}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right),$$

where the perturbations are given by a disturbing function R.

Gaussian equations of motion:

$$\frac{dp}{dt} = \frac{2pC}{w}\sqrt{\frac{p}{\mu}}$$

$$\frac{df}{dt} = \sqrt{\frac{p}{\mu}}\left\{S\sin(L) + \frac{[(w+1)\cos(L)+f]C}{w} - \frac{g(h\sin(L)-k\cos(L))N}{w}\right\}$$

$$\frac{dg}{dt} = \sqrt{\frac{p}{\mu}}\left\{-S\cos(L) + \frac{[(w+1)\sin(L)+g]C}{w} - \frac{f(h\sin(L)-k\cos(L))N}{w}\right\}$$

$$\frac{dh}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2N}{2w}\cos(L)$$

$$\frac{dk}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2N}{2w}\sin(L)$$

$$\frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \sqrt{\frac{p}{\mu}}\frac{(h\sin(L)-k\cos(L))N}{w},$$

where C, S, and N are components of the perturbing acceleration in the directions perpendicular to the radius vector in the direction of motion, along the radius vector outward, and normal to the orbital plane in the direction of the angular momentum vector, respectively.

Table 3 Availability of VOP equations of motion for selected element sets

Element set	Availability		Reference(s)
	Lagrange	Gauss	
(c) Modified classical	Yes	Yes	[5] (pp. 483, 488), [23] (pp. 308–311, pp. 314–316)
(g) Equinoctial	Yes	Yes	[20] [Eqs. (2.38a)–(2.38f)], [16] (p. 584, pp. 591–592)
(i) Alternate equinoctial	Yes	No	[9] (pp. 191, 192), [10] (pp. 454, 455)
(k) Modified equinoctial	Yes	Yes	[12] (pp. 412, 413), [13] (391, 392)
(m) Quaternion	No	Yes	[11] (p. 14)
(o) Euler parameters	Yes	Yes	[15] (p. 1082), [24] [p. 227; p. 228, Eq. (36)]
(q) Delaunay	No	No	N/A
(r) Poincaré	No	No	N/A

Table 4 Conversion between orbital elements and position and velocity vectors for selected element sets

Element set	Conversion to or from Cartesian coordinates	Reference(s)
(c) Modified classical	Yes	[2] (pp. 61–73), [15] (p. 1079)
(g) Equinoctial	Yes	[4] (pp. 3, 4), [6] (p. 309), [20] (pp. 85–88), [21] (p. 934 ff), [3] (p. 356 ff)
(i) Alternate equinoctial	No	N/A
(k) Modified equinoctial	No	N/A
(m) Quaternion	Yes	[11] (p. 11)
(o) Euler parameters	Yes (to)/no (from)	[15] (p. 1080)
(q) Delaunay	No	N/A
(r) Poincaré	Yes	[18] (pp. 80–83)

Conclusions

From the above survey, it is evident that there is significant liberty in the choice of a suitable set of state variables or elements. The modified equinoctial element set [Table 1(k)] is the only one that is nonsingular for all values of eccentricity and inclination. This set also employs elements that are not far from the classical ones, so that transforming and interpreting them in terms of physically significant parameters is relatively easier than using quaternions, for example. Therefore, it is advisable to use the modified equinoctial orbit element set for the research and technology development task [1]. The equinoctial elements can be used for the integration of orbits with special and general perturbations as well as differential corrections in orbit determination [6]. However, the other orbit element sets could prove to be convenient in specific applications where the singularities are not a problem.

Acknowledgments

The work described in this paper was carried out in part at the Jet Propulsion Laboratory, California Institute of Technology under the contract with the National Aeronautics and Space Administration. The author gratefully acknowledges the advice and suggestions of Paul J. Cefola, Anastassios E. Petropoulos, and Ryan P. Russell in the conduct of this survey.

References

[1] Petropoulos, A. E., Russell, R. P., and Hintz, G. R., “Feedback Control Laws for Preliminary Low-Thrust Trajectory Design in Multi-Body Regimes,” Jet Propulsion Laboratory, Annual Report, part of Research and Technology Program, JPL Task No. 01STCR R.06.023.020, Pasadena, CA, Sept. 2007.

[2] Bate, R. E., Mueller, D. D., and White, J. E., *Fundamentals of Astronautics*, Dover Publications, Inc., New York, 1971, pp. 58–60.

[3] Chobotov, V. A. (ed.), *Orbital Mechanics*, 3rd ed., AIAA Education Series, AIAA, Reston, VA, 2002, Chaps. 3, 14.

[4] Cefola, P. J., “Equinoctial Orbit Elements—Application to Artificial Satellite Orbits,” AIAA Paper 72-937, Sept. 1972.

[5] Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, AIAA, Reston, VA, 1999, Chap. 10.

[6] Broucke, R. A., and Cefola, P. J., “On the Equinoctial Orbit Elements,” *Celestial Mechanics*, Vol. 5, No. 3, May 1972, pp. 303–310. doi:10.1007/BF01228432

[7] Arsenault, J. L., Ford, K. C., and Koskela, P. E., “Orbit Determination Using Analytic Partial Derivatives of Perturbed Motion,” *AIAA Journal*, Vol. 8, No. 1, 1970, pp. 4–12.

[8] Brouwer, D., and Clemence, G., *Methods of Celestial Mechanics*, Academic Press, New York, 1961, pp. 241, 288.

[9] Giacaglia, G. E. O., “The Equations of Motion of an Artificial Satellite in Nonsingular Variables,” *Celestial Mechanics*, Vol. 15, No. 2, 1977, pp. 191–215. doi:10.1007/BF01228462

[10] Nacozy, P. E., and Dallas, S. S., “The Geopotential in Nonsingular Orbital Elements,” *Celestial Mechanics*, Vol. 15, No. 4, 1977, pp. 453–466. doi:10.1007/BF01228611

[11] Cohen, C. J., and Hubbard, E. C., “A Nonsingular Set of Orbital Elements,” *Astronomical Journal*, Vol. 67, No. 1, Feb. 1962, pp. 10–15. doi:10.1086/108597

[12] Walker, M. J. H., Ireland, B., and Owens, J., “A Set of Modified Equinoctial Orbit Elements,” *Celestial Mechanics*, Vol. 36, No. 4, 1985, pp. 409–419. doi:10.1007/BF01227493

[13] Walker, M. J. H., “A Set of Modified Equinoctial Orbit Elements,” *Celestial Mechanics and Dynamical Astronomy*, Vol. 38, No. 4, 1986, pp. 391, 392 (Erratum).

[14] Kustaanheimo, P., and Stiefel, E., “Perturbation Theory of Kepler Motion Based on Spinor Regularization,” *Journal fur die Reine und Angewandte Mathematik*, Vol. 218, 1965, pp. 204–219.

[15] Gurfil, P., “Euler Parameters as Nonsingular Orbital Elements in Near-Equatorial Orbits,” *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 5, Sept.–Oct. 2005, pp. 1079–1083.

[16] Vallado, D. A., *Fundamentals of Astrodynamics and Applications*, 2nd ed., Space Technology Library, Published jointly by Microcosm Press, El Segundo, CA, and Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001, Chaps. 2, 9.

[17] Smart, W. M., *Celestial Mechanics*, Longmans, Green and Co., New York, 1953, pp. 173–174.

[18] Lyon, R. H., “Geosynchronous Orbit Determination Using Space Surveillance Network Observations and Improved Radiative Force Modeling,” M.S. Thesis, Massachusetts Institute of Technology, Cambridge, MA, June 2004.

[19] Kamel, A. A., “New Nonsingular Forms of Perturbed Satellite Equations of Motion,” *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 5, Sept.–Oct. 1983, pp. 387–392.

[20] McClain, W. D., “A Recursively Formulated First-Order Semi-Analytic Artificial Satellite Theory Based on the Generalized Method of Averaging,” Vol. 1 (The Generalized Method of Averaging Applied to

the Artificial Satellite Problem), Computer Sciences Corporation, CSC/TR-77/6010, Silver Spring, MD, Nov. 1977 (most recent update in 1992).

[21] Kechichian, J. A., "Equinoctial Orbit Elements: Application to Optimal Transfer Problems," AIAA Paper 90-2976, Aug. 1990.

[22] Van der Ha, J. C., "Non-Singular and Non-Conventional Orbit Perturbation Equations," *Zeitschrift fur Flugwissenschaften und Weltraumforschung*, Vol. 9, No. 4, 1985, pp. 217–224.

[23] Taff, L. G., *Celestial Mechanics: A Computational Guide for the Practitioner*, Wiley, New York, 1985, pp. 308–316.

[24] Gurfil, P., and Klein, I., "Nonsingular Modeling of Planetary Equinoctial Precession Using Euler Parameters," *Planetary and Space Science*, Vol. 55, Nos. 1–2, Jan. 2007, pp. 223–236. doi:10.1016/j.pss.2006.06.003